However, he assumed that the maximum effect would be produced by a totally black disc, and it is not clear that this is true. If we add only inelastic states to the model, however, we find that we can produce a P-wave resonance virtually anywhere. The *unique* value for the position and width of the ρ meson is obtained when one imposes crossing symmetry as well. Thus it would appear that all aspects are equally important; elastic unitarity and the correct crossed cuts yield a dasically repulsive interaction, the attraction necessary to produce a resonance is provided by the inelastic states (a phenomenon observed in other calculations[7]), and the actual value of the resonance is determined by the crossing relations.

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Asymptotic Behavior of Partial-Wave Amplitudes

R. Omnes
Lawrence Radiation Laboratory, University of California, Berkeley, California

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For infinite energies, we determine the asymptotic behavior of partial-wave amplitudes when the full scattering amplitude satisfies Mandelstam representation and has itself a Regge asymptotic behavior. Particular attention is paid to the behavior of the partial-wave-amplitude discontinuities on their cuts. They are shown to behave as $|t|^{\alpha(0)-1}$, where t is the energy squared and $\alpha(0)$ is the leading Regge-pole position at zero energy. This result removes an old-standing difficulty in the Chew-Mandelstam calculation of amplitudes and provides a precise justification of the nearest singularity technique. As an application, we show that no subtraction is necessary in partial-wave-amplitude dispersion relations at physical values of the angular momentum, even for the case of S waves.

I. INTRODUCTION

In their original program, Chew and Mandelstam stressed that a particle or a resonance in a crossed channel contributes to the forces acting between two particles.[1] More precisely, the partial-wave amplitudes for pion-pion scattering have both a left- and a right-hand cut as functions of the energy, and the resonances in the crossed channels determine the discontinuity across the left-hand cut or, equivalently, the forces. Unfortunately, it appeared that the discontinuity obtained from that mechanism increased at a rate in conflict with unitarity when the energy became infinite and negative, as soon as the spin of the resonance or of the bound state in the crossed channel was larger than or equal to one. Such is the case for the ρ meson (and now also for the f^0 meson). The problem of determining the exact high-energy behavior of amplitudes became a necessary preliminary to the dispersion theory of elementary particles.

It was indeed felt that a simple solution of the problem had to exist since, in several cases, the simple trick of introducing a cutoff for the left-hand cut discontinuity leads to sensible results. This idea has been expressed as the nearest singularity hypothesis, by which one meant that a physical process was mostly determined by the effects of the singularities nearest to the physical region and was not affected by any misbehavior of the amplitudes at infinity.[2]

The clue to a solution of the problem was provided by the observation, due to Regge,[3] that the asymptotic behavior of the nonrelativistic-scattering amplitudes, as functions of the angle, are determined by the singularities of the partial-wave amplitudes as functions of a continuous angular momentum.[3] Actually, these singularities are only poles. Chew and Frautschi[4] and Mandelstam[5] pointed out that the high-energy difficulties of the S-matrix theory of strong interactions could be eliminated if one takes as an ansätz that the asymptotic behavior of the total amplitude in relativistic theory is analogous to the one found in nonrelativistic theory.

Although it was clear that the asymptotic difficulties were removed by that hypothesis, one had yet to exhibit a practical way of resuming the Chew-Mandelstam program, now enlarged to be a program for self-consistently computing the leading Regge-pole trajectories. Chew and Jones are currently investigating such an approach in which they work both with the full amplitude and with the partial-wave amplitudes.[6,7] However, it is not clear whether only using the partial-wave amplitudes, which has the advantage of leading to one-dimensional well-known equations, could lead

[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] See, for instance, G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961).
[3] T. Regge, Nuovo Cimento **18**, 947 (1960).
[4] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[5] S. Mandelstam (unpublished).
[6] G. F. Chew and C. E. Jones, Lawrence Radiation Laboratory Report UCRL-10992, August 1963 (unpublished).
[7] G. F. Chew, Conferences at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England, 1963 (unpublished); see also Ref. 9.

to simpler or more accurate calculations. Before we enter into such a program, nevertheless, it is necessary to solve explicitly the preliminary problem of high-energy divergences and to find the explicit behavior of partial-wave amplitude discontinuities once the Regge asymptotic behavior is assumed for the full amplitude.

In the present paper, we devote our work to the solution of this problem. Our main result is that the discontinuity at infinity is determined by the position of the leading Regge pole at zero total energy $\alpha(0)$ and not by the spin of the physical bound states or resonances. Since, as has been shown by Froissart,[8] the requirement of unitarity implies that $\alpha(0)\leqslant 1$, it may be shown that the difficulty originally encountered by Chew and Mandelstam is removed. In fact, the discontinuities on the left-hand cut and the right-hand cut fit so well that no subtraction is needed in any physical partial-wave dispersion relation, even for the S wave. Therefore, no subtraction parameters have to be introduced when one solves the Chew-Mandelstam equations.

Apart from this result, which bears on the consistency of the theory, it will be useful to use the asymptotic form of the discontinuities in such practical applications as the determination of the Regge-pole trajectory from N/D equations. In fact, it is obvious that in the particular case of the S wave this procedure will lead to more rapidly convergent calculations than those which can be obtained by introducing a cutoff.

The simple form of the discontinuity, which is not oscillating as sometimes assumed,[9] but smoothly damped, is in fact a justification of the nearest singularity method.

After some preliminaries about partial-wave amplitudes, their properties, and a precise statement of our hypotheses in Secs. I to III, we compute the asymptotic behavior of the discontinuities in Secs. IV through VII. Applications to the number of subtractions in partial-wave-dispersion relations, as well as the possibility of cuts in the angular momentum plane, are made in Sec. VIII.

II. HYPOTHESES

Let us consider the amplitude $A(s,t)$ for the scattering of identical neutral spinless particles with mass unity. We consider t to be the total center-of-mass energy squared, and s to be the square of the invariant momentum transfer. The partial-wave-scattering amplitudes $a_l(t)$ are given by

$$a_l(t)=\frac{1}{2}\int_{-1}^{+1} A(z,t)P_l(z)dz\,, \tag{1}$$

where $z=1+2s/(t-4)$, is the cosine of the c.m. scattering angle. When $A(t,z)$ is an analytic function of z in a domain that contains the segment $(-1,+1)$, Eq. (1) can be replaced by the Neumann formula

$$a_l(t)=\frac{1}{2\pi i}\int_C A(z,t)Q_l(z)dz\,, \tag{2}$$

where C is a contour around -1 and $+1$. In particular, when $A(z,t)$ is analytic in the complex z plane cut from z_0 to ∞, as is the case for any nonrelativistic amplitude satisfying the Mandelstam representation, Eq. (2) can be replaced by

$$a_l(t)=\frac{1}{\pi}\int_{z0}^{\infty} Q_l(z)A_s(s,t)dz\,, \tag{3}$$

where $2iA_s(s,t)$ is the discontinuity of $A(s,t)$ across its s cut. Equation (3) was first given by Froissart.[10] Its main properties are that it can be extended to complex values of l, and that the function obtained in this way is analytic in at least a half-plane $\mathrm{Re}l>N$, where N is the necessary number of subtractions in a dispersion relation at the fixed energy t. Within that half-plane, $a_l(t)$ is a bounded function. In accordance with a theorem by Carlson,[11] it is therefore the unique analyitc function that interpolates the physical amplitudes from l integer to l complex and that does not at most increase as fast as $\exp\pi|l|$. Equation (1) does not verify this boundedness property and coincides with Eq. (2) only for positive integer values of l.

For a relativistic amplitude satisfying the Mandelstam representation, there are two cuts, and Eq. (3) has to be replaced by

$$a_l(t)=\frac{1}{\pi}\int_{z0}^{\infty} Q_l(z)A_s(s,t) +\frac{(-1)^l}{\pi}\int_{z0}^{\infty} Q_l(z')A_u(u,t)dz'\,, \tag{4}$$

so that one has to deal with two different analytic functions of l that coincide respectively with $a_l(t)$ for even and odd values of l. One distinguishes these two analytic functions by their signature ± 1. For simplicity, we shall only consider in this paper the simple model of neutral spinless particles for which the odd amplitude vanishes and only the amplitudes with even signature play a role.

III. DISPERSION RELATION

The function $a_l(t)$ has a branch point at the two-body threshold $t=\mu$ and it is convenient to define, with Gribov,[12] the new function $\phi_l(t)=a_l(t)/(t-\mu)^l$, which

[8] M. Froissart, Phys. Rev. **123**, 1053 (1961).

[9] G. F. Chew, Phys. Rev. **129**, 2363 (1963).

[10] M. Froissart, International Conference on Strong Interactions, La Jolla, California, 1961 (unpublished).

[11] R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1956).

[12] V. N. Gribov, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962).

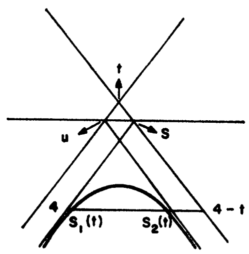


FIG. 1. Bound of the integral for $\Delta\phi^{(2)}$.

has only a second-order branch point at the two-body threshold. The function $\phi_l(t)$ satisfies the dispersion relation, for Rel large enough and we have

$$\phi_l(t)=\frac{1}{\pi}\int_4^\infty \frac{\mathrm{Im}\phi_l(t')dt'}{t'-t}+\frac{1}{\pi}\int_{-\infty}^0 \frac{\Delta\phi_l(t')dt'}{t'-t}, \quad (5)$$

where it is understood that the necessary number of subtractions have been made. When Rel becomes small enough, it is necessary to add to Eq. (5) the contribution of the poles of $\phi_l(t)$ (which are at the position of bound states when l is a positive integer or zero).

The discontinuities on the cuts are given, for t positive, by

$$\mathrm{Im}\phi_l(t)=\frac{4}{\pi}\int_4^\infty Q_l\left(1+\frac{2s}{t-4}\right)A_{st}(s,t)\frac{ds}{(s-4)^{l+1}}. \quad (6)$$

Here A_{st} is the Mandelstam weight function $\rho(s,t)$. For negative t, one has

$$\Delta\phi_l(t)=\Delta\phi_l^{(1)}(t)+\Delta\phi_l^{(2)}(t), \quad (7a)$$

$$\Delta\phi_l^{(1)}(t)=2\int_4^{4-t} P_l\left(\frac{2s}{4-t}-1\right)A_s(s,t-i\epsilon)\frac{ds}{(4-t)^{l+1}}, \quad (7b)$$

and

$$\Delta\phi_l^{(2)}(t)=\frac{4}{\pi}\int_{s_1(t)}^{s_2(t)} Q_l\left(\frac{2s}{4-t-i\epsilon}-1\right)A_{su}(s,t)\frac{ds}{(4-t)^{l+1}}, \quad (7c)$$

where $s_1(t)$ and $s_2(t)$ are the boundary of the third spectral region for fixed t (see Fig. 1). Note that, while $\Delta\phi_l(t)$ is real, this is not true of $\Delta\phi_l^{(1)}$ and $\Delta\phi_l^{(2)}$ separately. It has been shown by Gribov and Pomeranchuk[13] that, due to the existence of poles of Q_l as functions of l at $l=-1, -2, \cdots$, Eq. (7c) implies the existence of an essential singularity of $\phi_l(t)$ at $l=1$.

IV. BEHAVIOR ON THE RIGHT-HAND CUT

Our problem is to investigate the behavior of the three discontinuities given by Eqs. (7) when t tends towards $\pm\infty$. To do this, we make the following assumptions:

(a) The amplitude $A(s,t)$ satisfies the Mandelstam representation.

(b) When one of the variables s, t, or u tends to infinity, the asymptotic behavior of the amplitude is of the Regge type.[14] More precisely, if t tends to infinity at a fixed value of s, one has

$$A(s,t)\approx\sum_r \gamma_r(s) \times\frac{P_{\alpha_r(s)}[-1-(2t/(s-4))]+P_{\alpha_r(s)}[1+(2t/(s-4))]}{\sin\pi\alpha_r(s)}, \quad (8)$$

where the sum goes over the indices r of the Regge-poles trajectories. We assume explicitly that there are no cuts in the angular-momentum plane. For simplicity, we make all the subsequent considerations by taking only into account one Regge pole.

(c) We suppose that $\alpha_r(s)$ is an analytic function of s in the complex s plane cut from $s=4$ to $+\infty$, and that it has a limit when α tends to infinity so that by the formal transformation defined by $\alpha_r(s)$, it transforms the s plane into the kidney-shaped region indicated in Fig. 2.

In fact, it would be easy to trace out the modifications of the following arguments if some of these assumptions were to fail. They are just made here for the sake of simplicity.

Again, for more clarity, we do not consider the Regge poles of the u channel, but only those of the s channel as they are exhibited in Eq. (8). This is equivalent to putting the residues of the u-channel Regge poles identically equal to zero and taking into account only the s-channel Regge poles, then exchanging the role of s and u and adding the results. That simply makes the equations shorter so that when we reestablish the contributions of the u-channel Regge poles at the end of the argument, we shall only have to multiply in some places by a factor of 2.

Lastly, we insist on the reality of $\Delta\phi_l(t)$ for l—real by writing, in place of Eq. (7),

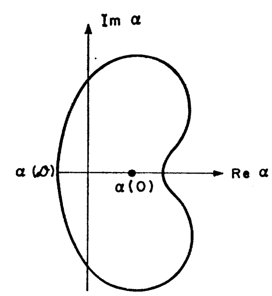
$$\Delta\phi_l(t)=\Delta\phi_l'^1(t)+\Delta\phi_l'^2(t), \quad (9)$$

where

$$\Delta\phi_l'^1(t)=\mathrm{Re}\Delta\phi_l^{(1)}(t) \text{ and } \Delta\phi_l'^2(t)=\mathrm{Re}\Delta\phi_l^{(2)}(t). \quad (10)$$

Let us first find the asymptotic behavior of the dis-

FIG. 2. Image of the cut s plane by the $\alpha(s)$ conformal transformation.



^{13} V. N. Gribov and I. Ya. Pomeranchuck, Phys. Letters 2, 239 (1962).

^{14} S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. 126, 2206 (1962).

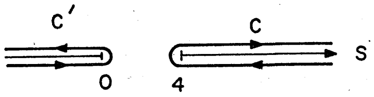


FIG. 3. Singularities of the integrand in Eq. (11).

continuity on the right-hand cut. This behavior may be found in a variety of ways and the result is already well known since it is the behavior of the phase shifts when the diffraction peak is determined by a Regge pole. However, we derive it in a way that is a good illustration of the method to be employed for the other discontinuities.

As we do not take into account the u-channel Regge poles, which when t tends to infinity, is equivalent to considering the A_{tu} discontinuity as zero, Eq. (6) reads now

$$\mathrm{Im}\phi_l(t)=\frac{2}{\pi}\int_4^\infty Q_l\left(1+\frac{2s}{t-4}\right)A_{st}(s,t)\frac{ds}{(t-4)^{l+1}}. \quad (11)$$

Let us write it as a contour integral

$$\mathrm{Im}\phi_l(t)=\frac{1}{i\pi}\int_C Q_l\left(1+\frac{2s}{t-4}\right)A_t(s,t)\frac{ds}{(t-4)^{l+1}}, \quad (12)$$

where the contour C goes around the cut of $A_t(s,t)$ from $s=4$ to $s=\infty$ (see Fig. 3). The integrand in Eq. (12) has this cut and also has the cut of the Legendre function which goes from $s=-\infty$ to $s=0$. If we make the conformal transformation from the s plane to the α plane, Eq. (12) remains true as an integral over α on the contour C shown in Fig. 4. The contour C can be reduced to C', which encloses the Q_l cut. It is clear from Fig. 4 that an integral over C' cannot increase more strongly than $t^{\alpha(0)}$ when t tends to infinity.

For the part of C' that is in the physical region (i.e., for $4-t<s<0$), one can compute the discontinuity of C' by using the relation[15]

$$Q_l(x-i\epsilon)-Q_l(x+i\epsilon)=i\pi P_l(x), \quad (13)$$

which is true for $-1\leqslant x\leqslant 1$. However, it is clear from Fig. 4 that, as t tends to infinity, the part of the integration over C' for $s<4-t$ behaves as $t^{\alpha(\infty)}$. If we neglect

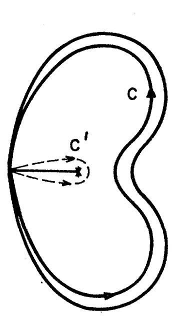


FIG. 4. Deformation of a contour leading to the asymptotic behavior of $\mathrm{Im}\phi_l(t)$.

[15] Bateman's Manuscript Project, in *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1950).

such corrections of the order $t^{\alpha(\infty)}$, we thus get

$$\mathrm{Im}\phi_l(t)\approx\int_{-\infty}^0 P_l\left(1+\frac{2s}{t-4}\right)A_t(s,t)\frac{ds}{(t-4)^{l+1}} +0[t^{\alpha(\infty)-l-1}]. \quad (14)$$

Obviously, this result can be obtained without using the conformal transformation. However, this transformation is extremely useful in the next two cases. It shows immediately what the asymptotic behavior is of any contour integral just by indicating up to what point in the α plane the contour can be pushed to the left, as well as indicating the asymptotic behavior of any term to be neglected.

V. BEHAVIOR ON THE LEFT-HAND CUT

Let us now look for the asymptotic form of $\Delta\phi_l^{(1)}(t)$. If we take into account only the s cut, Eq. (7b) reads

$$\Delta\phi_l^{(1)}(t)=\int_4^{4-t} P_l\left(\frac{2s}{4-t}-1\right)A_s(s,t)\frac{ds}{(4-t)^{l+1}}. \quad (15)$$

The same argument as for $\mathrm{Im}\phi_l(t)$ can now be given.

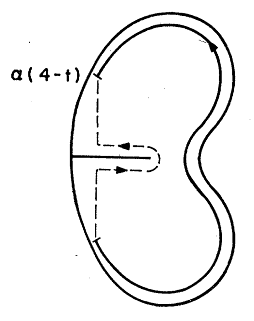


FIG. 5. Deformation of contour for $\Delta\phi_l^{(1)}$.

First transform Eq. (15) into

$$\Delta\phi_l^{(1)}=\frac{1}{2i}\int_{C_1} P_l\left(\frac{2s}{4-t}-1\right)A(s,t)\frac{ds}{(4-t)^{l+1}}, \quad (16)$$

where the path C_1 goes from $4-t-i\epsilon$ to $4-t+i\epsilon$ and encloses the branch point at $s=4$. Here again we can push the transformed path C_1 to the left in the α plane (see Fig. 5) and replace C_1 by the preceding contour C'. The error committed is caused by that part of the integration path in Fig. 5 going from $\alpha(4-t)$ to $\alpha(\infty)$, which gives terms of the order of $t^{\alpha(\infty)}$ for t large enough. By using now the relation[15]

$$P_l(x-i\epsilon)-P_l(x+i\epsilon)=-2i\sin\pi l P_l(-x) \quad (17)$$

(which is valid for $x<-1$), to compute the integral on C', we get

$$\Delta\phi_l^{(1)}(t)=-\sin l\pi\int_{-\infty}^0 P_l\left(1-\frac{2s}{4-t}\right)A(s,t) +0[t^{\alpha(\infty)-l-1}]. \quad (18)$$

VI. CALCULATION OF $\Delta\phi_l^{(2)}(t)$

The calculation of $\Delta\phi_l^{(2)}(t)$ is exactly of the same type as in the preceding section, although more involved. As a first step we define the new function

$$B(s,t)=\frac{1}{\pi}\int_{s_1(t)}^{s_2(t)}\frac{A_{su}(s',t)ds'}{s'-s}. \qquad (19)$$

It is important to observe that $B(s,t)$ behaves asymptotically as $A_u(s,t)$ when t tends to $-\infty$, s being kept fixed and negative. In order to show this, we observe that $A_{su}(s',t)$ behaves, when $|t|\rightarrow\infty$, as

$$A_{su}(s,t)\sim\mathrm{Im}\gamma(s)|t|^{\alpha(s)}, \qquad (20)$$

and we write the asymptotic form of $B(s,t)$ as

$$\frac{1}{\pi}\int_4^{\infty}+\int_{s_1(t)}^{4}+\int_{\infty}^{s_2(t)}\frac{\mathrm{Im}\gamma(s')|t|^{\alpha(s')}}{s'-s}ds'. \qquad (21)$$

It is easy to show that the last integral behaves as $|t|^{\alpha(\infty)-1}$ when t tends to $-\infty$ and that the second integral behaves as $|t|^{\alpha(4)-3/2}$. [The integrand behaves

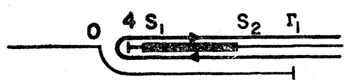


FIG. 6. Singularities of the integrand for $\Delta\phi_l^{(2)}$, s plane.

as $|t|^{\alpha(4)}$, the integration interval from 4 to $s_1(t)$ is proportional to t^{-1}, and a more careful examination of the effect of the branch point of $\alpha(s)$ at $s=4$ leads to the last $|t|^{-1/2}$ factor.] Therefore, up to powers of t smaller than $\alpha(4)-3/2$ or $\alpha(\infty)-1$, $B(s,t)$ is given asymptotically by the first integral, which can be written as

$$\frac{1}{2\pi i}\int_C\frac{\gamma(s')|t|^{\alpha(s')}}{s'-s}ds',$$

where, again, C is a contour surrounding the cut of $\alpha(s)$ and $\gamma(s)$ from 4 to ∞. By the Cauchy theorem, this is precisely $\gamma(s)|t|^{\alpha(s)}$; i.e., we have shown that

$$B(s,t)\approx A_u(s,t)+0[|t|^{\mathrm{Max}\,\alpha(\infty)-1,\alpha(4)-3/2}], \qquad (22)$$

when t tends to $-\infty$. We rewrite Eq. (7c) as

$$\Delta\phi_l^{(2)}(t)=\frac{4}{2\pi i}\int_{\Gamma_1}Q_l\left(\frac{2s}{4-t-i\epsilon}-1\right)B(s,t)\frac{ds}{(4-t)^{l+1}}, \qquad (23)$$

where the contour Γ_1 has to surround the cut of $B(s,t)$ from $s_1(t)$ to $s_2(t)$ and must avoid the cuts of the Legendre functions. The Legendre function has two logarithmic singular points at $s=0$ and $s=4-t-i\epsilon$, and it is customary to join these two points to $s=-\infty$ by a common cut as shown in Fig. 6. As for the contour Γ_1, we shall choose it as shown in Fig. 6 by making it go from $+\infty$ and back by turning around $s=4$.

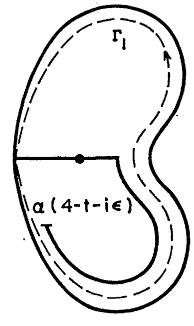


FIG. 7. Singularities of the integrand for $\Delta\phi_l^{(2)}$, α plane.

To find the asymptotic behavior of $\Delta\phi_l^{(2)}(t)$, we proceed by the following steps:

(a) Make the conformal transformation from s to $\alpha(s)$ (see Fig. 7).

(b) Split the Q_l cut into its two component cuts from $\alpha(\infty)$ to $\alpha(4-t)$ and from $\alpha(\infty)$ to $\alpha(0)$.

(c) Apply the contour Γ_1 against these cuts.

(d) Deform the cut which goes from $\alpha(\infty)$ to $\alpha(4-t)$, along with the contour which encloses it, to push the whole pattern as much as possible to the left. One is then led to the situation shown in Fig. 8.

(e) The contour around the cut from $\alpha(\infty)$ to $\alpha(4-t)$ gives a contribution to $\Delta\phi_l^{(2)}$, which behaves asymptotically as $t^{\alpha(\infty)}$; we drop it, keeping only the contour which encloses the cut from $\alpha(\infty)$ to $\alpha(0)$.

(f) Making the conformal transformation from α to s, we see that, asymptotically, $\Delta\phi_l^{(2)}$ is equal to

$$\Delta\phi_l^{(2)}(t)=\frac{4}{2\pi i}\int_{\Gamma_2}Q_l\left(\frac{2s}{4-t-i\epsilon}-1\right)B(s,t)\frac{ds}{(4-t)^{l+1}}, \qquad (24)$$

where Γ_2 encloses the cut of the Legendre function from $s=-\infty$ to $s=0$, as shown in Fig. 9. Therefore, taking Eqs. (10) and (22) into account, we get

$$\Delta\phi_l'^{(2)}(t)=\mathrm{Re}\frac{4}{2\pi i}\times\int_{\Gamma_2}Q_l\left(\frac{2s}{4-t-i\epsilon}-1\right)B(s,t)\frac{ds}{(4-t)^{l+1}}. \qquad (25)$$

To find the discontinuity of the Legendre function along the cut of interest one uses Eq. (13), which gives

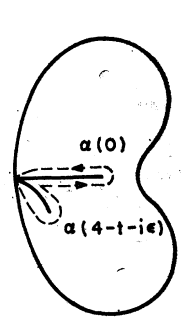


FIG. 8. Deformation of cuts and contours for $\Delta\phi_l^{(2)}$, α plane.

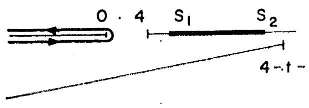


FIG. 9. Deformation of cuts and contours for $\Delta\phi_l^{(2)}$, s plane.

the discontinuity of Q_l across the other cut, one also uses[15]

$$Q_l(-z)=-e^{\pm il\pi}Q_l(z)\,, \tag{26a}$$

and

$$P_l(-z)=e^{\mp il\pi}P_l(z)-\frac{2}{\pi}\sin\pi l Q_l(z)\,, \tag{26b}$$

where Eq. (26a) gives the discontinuity across the merged cuts and (26b) serves to evaluate the right-hand member of Eq. (13). Finally, we have

$$\Delta\phi_l'^{(2)}(t)=\cos l\pi\int_{-\infty}^{0}P_l\left(1-\frac{2s}{4-t}\right)A_u(s,t)\frac{ds}{(4-t)^{l+1}}+0[t^{\alpha(\infty)-l-1}]. \tag{27}$$

VII. RESULTS

Let us now summarize the asymptotic values of the discontinuities when due care is taken of both the s and u Regge poles. This leads to

$$\mathrm{Im}\phi_l(t)=2\int_{-\infty}^{0}P_l\left(1+\frac{2s}{t-4}\right)A_t(s,t)\frac{ds}{(t-4)^{l+1}}+0[t^{\alpha(\infty)-l-1}], \tag{28a}$$

and

$$\Delta\phi_l(t)=2\cos l\pi\int_{-\infty}^{0}P_l\left(1-\frac{2s}{4-t}\right)A_u(s,t)\frac{ds}{(4-t)^{l+1}}-2\sin l\pi\,\mathrm{Re}\int_{-\infty}^{0}P_l\left(1-\frac{2s}{4-t}\right)A(s,t)\frac{ds}{(4-t)^{l+1}}+0[|t|^{\alpha(\infty)-l-1}]. \tag{28b}$$

Replacing the Legendre function by its limit (equal to 1 when t tends to infinity), we get the less accurate result

$$\mathrm{Im}\phi_l(t)=2\int^{0}A_t(s,t)\frac{ds}{(t-4)^{l+1}}+0[t^{\alpha(0)-l-2}] \tag{29a}$$

and

$$\Delta\phi_l(t)=2\cos l\pi\int^{0}A_u(s,t)\frac{ds}{(4-t)^{l+1}}-2\sin l\pi\times\int^{0}\mathrm{Re}A(s,t)\frac{ds}{(4-t)^{l+1}}+0[|t|^{\alpha(0)-l-2}]. \tag{29b}$$

It is now obvious that both discontinuities behave as $t^{\alpha(0)-l-1}$, up to logarithmic factors, and that they are both damped without oscillations.

In fact, it is clear that the same method may be applied directly to the Froissart formula [Eq. (3)] and that the whole function $\phi_l(t)$ itself behaves as $t^{\alpha(0)-l-1}$ when $|t|$ tends to infinity.

VIII. APPLICATIONS

An important application of Eqs. (29) is to show that all dispersion relations for physical partial-wave amplitudes can be written without any subtraction. In fact, as can be seen from the Froissart theorem, one has $\alpha(0)\leqslant 1$, so that it is clear that both integrals in the dispersion relation (5) converge when $l>0$.

The case of $l=0$ has to be treated more carefully. Actually, an immediate consequence of Eqs. (29) is

$$\mathrm{Im}\phi_0(t)-\Delta\phi_0(t)=0(|t|^{-1})\,, \tag{30}$$

when we have $t\to\infty$. Therefore, if we write the dispersion relation in the form

$$\phi_0(t)=\frac{1}{\pi}\int_0^{\infty}\left[\frac{\mathrm{Im}\phi_0(t')}{t'-t}-\frac{\Delta\phi_0(-t')}{t'+t}\right]dt, \tag{31}$$

it is clear that this integral is rapidly convergent.

This result shows that, in fact, calculations involving the exact asymptotic behavior of partial-wave amplitudes will converge more rapidly than calculations where the left-hand cut contribution is cut off.

Another application demonstrates the relevance of partial-wave asymptotic behavior in the discussion of angular-momentum cuts. One can easily show, by introducing the asymptotic behavior

$$A_t(s,t)\sim\beta(s)t^{\alpha(s)}\,, \tag{32a}$$

$$A_u(s,t)\sim\beta(s)t^{\alpha(s)}\,, \tag{32b}$$

and

$$\mathrm{Re}A(s,t)\sim\beta(s)t^{\alpha(s)}\frac{(1+\cos\pi\alpha)}{\sin\pi\alpha} \tag{33}$$

into Eqs. (28), and then introducing the discontinuities into the dispersion relation Eq. (5), that each of the dispersion integrals in Eq. (5) is an analytic function of l with a cut going from $l=-\infty$ to $l=\alpha(0)-1$. However, if we examine the singularity at $\alpha(0)-1$, using Eqs. (32), we find that the singularity is cancelled if both dispersion integrals are considered together. This is another example where a correct account of the left-hand cut discontinuity at infinity gives much better results than a cutoff.

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